Quadrature

A tour through
Greek mathematical thought.

Adapted from <u>Journey Through Genius</u> by William Dunham

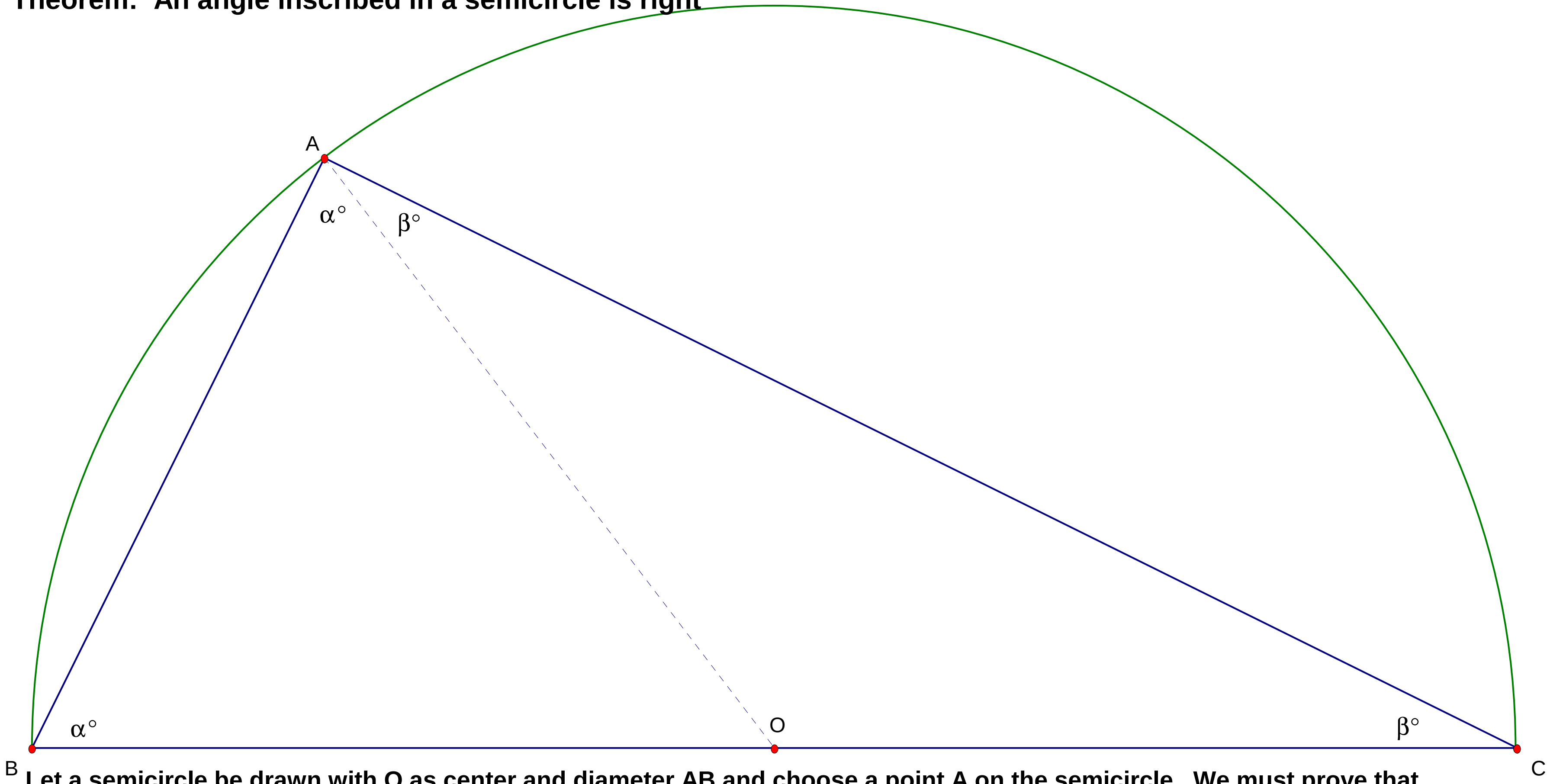
The appearance of Demonstrative Mathematics

- Egypt 2000 B.C.E.
  - Use but do not understand Pythagorean Triples
- Babylon 1600 B.C.E.
  - Understand but do not prove the Pythagorean Theorem
- Greece 600 B.C.E.
  - Thales and the beginning of Proof

Thales of Miletus <u>proved</u> that:

- Vertical angles are equal
- The angle sum of a triangle equals two right angles
- The base angles of an isosceles triangle are equal
- An angle inscribed in a semicircle is a right angle

Theorem: An angle inscribed in a semicircle is right



Let a semicircle be drawn with O as center and diameter AB and choose a point A on the semicircle. We must prove that $\angle BAC$ is right. Draw line OA and consider ΔAOB. Since $\overline{OA}$ and $\overline{OB}$ are radii of a circle, they are congruent and ΔAOB is isosceles. Hence, $\angle OAB \cong \angle OBA$. Call them both α. Similarly, in ΔAOC, $\angle OAC \cong \angle OCA$. Call them β.

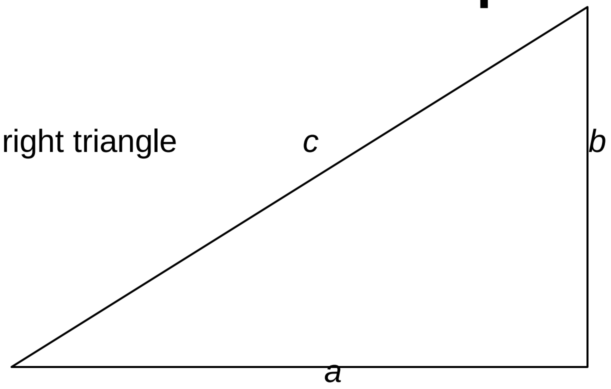
In large triangle BAC, 2 right angles = $\angle ABC + \angle BCA + \angle CAB = \alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta)$

Hence, one right angle = $\frac{1}{2}$(2 right angles) = $\frac{1}{2}2(\alpha + \beta) = \alpha + \beta = \angle BAC$. **QED**

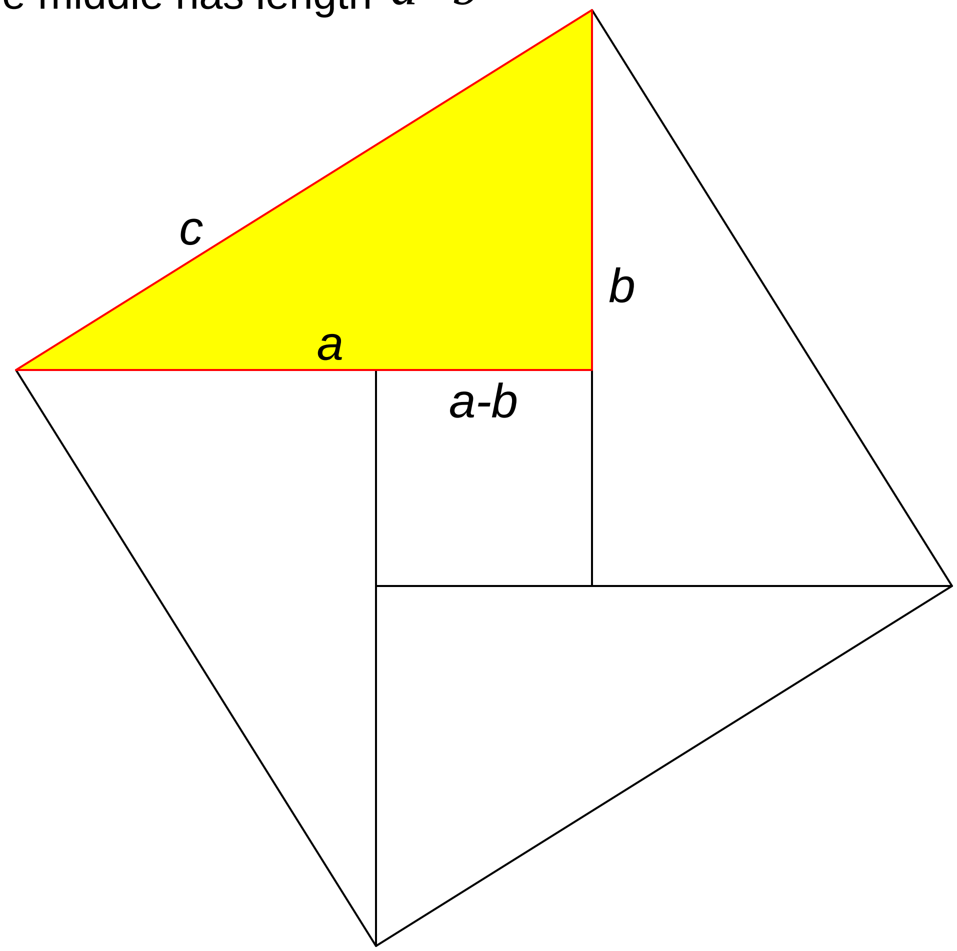
Pythagorus born 572 B.C.E

- Proves the Pythagorean theorem
- We offer here a sample proof

Consider a right triangle



Now, consider four of these triangles, arranged like this. Note that the small square in the middle has length $a-b$



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Clearly, the area of the large square is c^2

But the area of the large square is also equal to the sum of the area of the small square plus the area of the four triangles.

The area of the small square is $(a-b)^2 = a^2 - 2ab + b^2$

The area of each triangle is $\frac{1}{2}ab$

So the area of the four triangles is

$$4\left(\frac{1}{2}\right)ab = 2ab$$

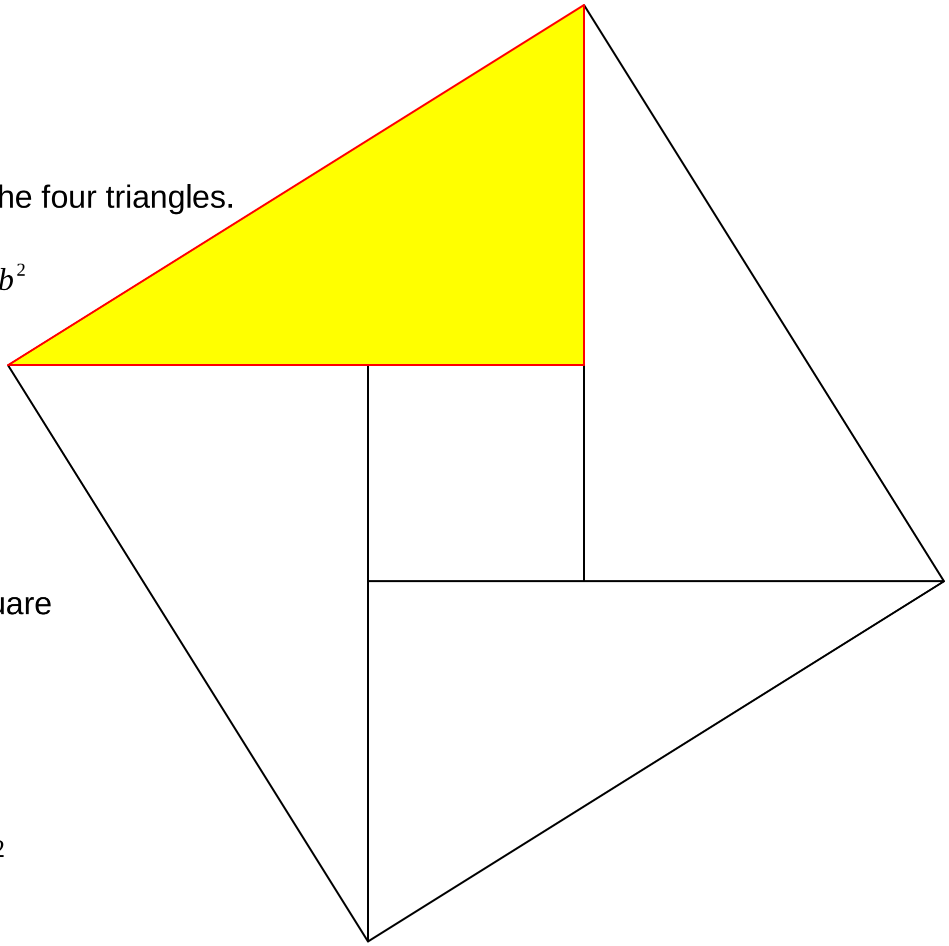
Now we see that the sum of the area of the small square plus the areas of the four triangles is

$$a^2 - 2ab + b^2 + 2ab = a^2 + b^2$$

which is also the area of the large square.

But we said that the area of the large square was c^2

. So, $c^2 = a^2 + b^2$ which is what we set out to prove.

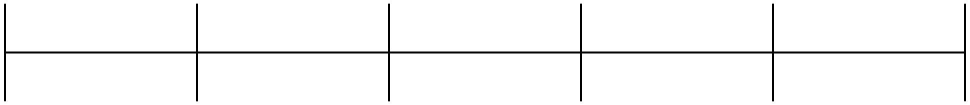


Pythagoreans

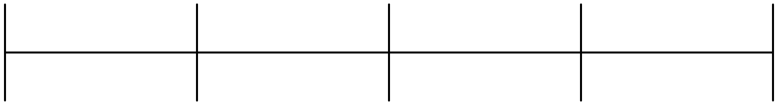
- All is number
- Ratios of numbers appear in:
  Geometry, Music, Astronomy
- The modern notion of the mathematization of science
- Hippasus discovers incomensurable numbers.
- The Pythagoreans throw him overboard.

AB and CD are comensurable because EF divides both

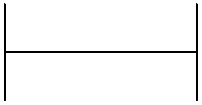
- A B



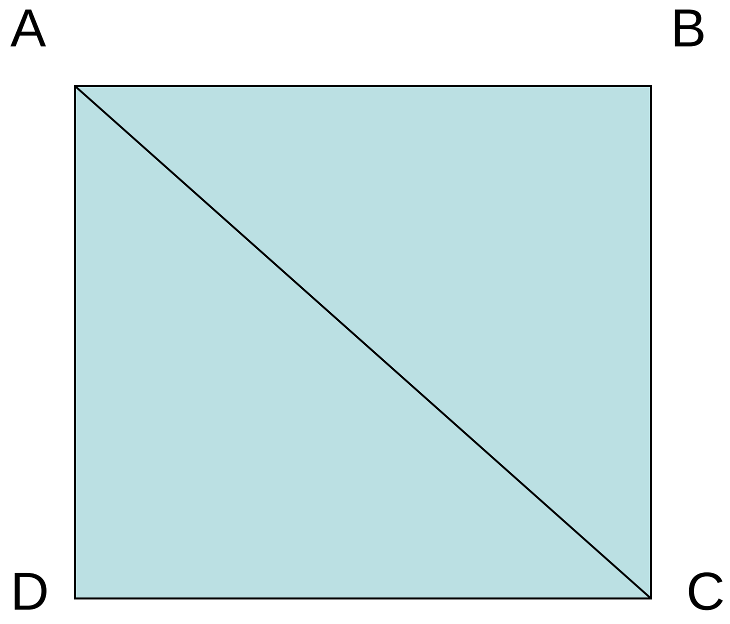
- C D



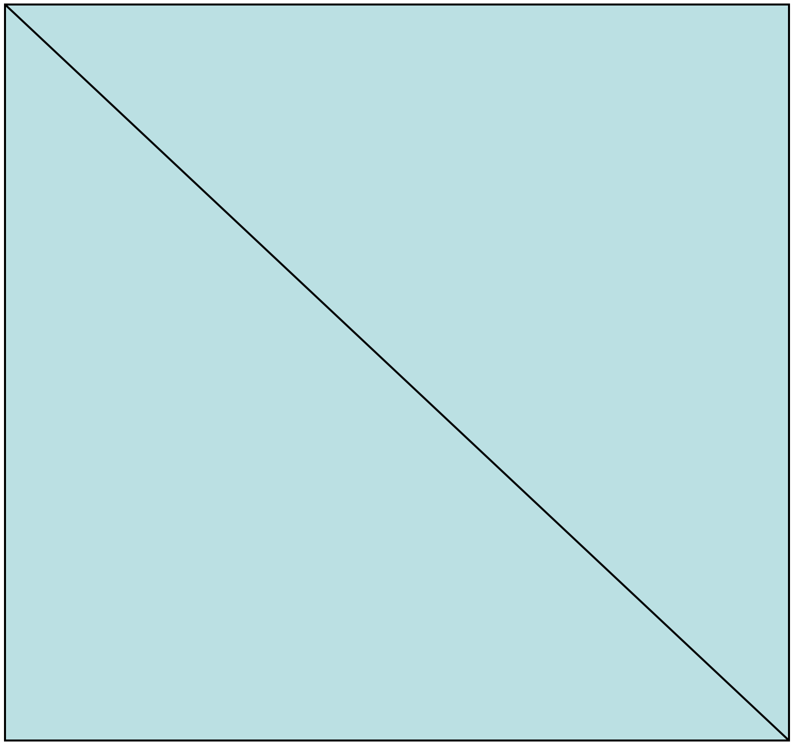
- E F



Square ABCD is above suspicion as a geometric object, but as numbers, AB and AC are problematic



No matter how small a unit is chosen, it is impossible to measure both the side and the diagonal at the same time.



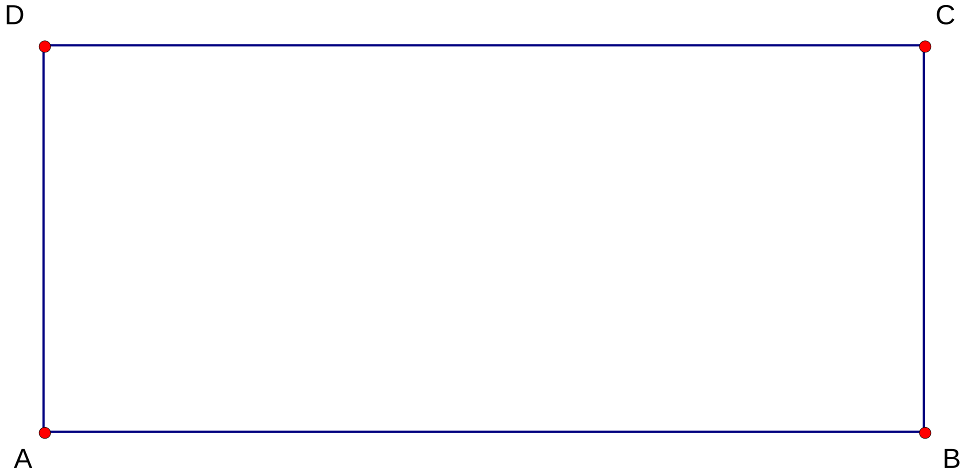
Quadrature

- Greeks loved symmetry and order
- Numbers and measurement were suspect
- Straightedge and compass represent the simplest one-dimensional figure (the line) and the simplest two-dimensional figure (the circle).
- To understand the area of a figure, they made a square with the same area

The Quadrature of the Rectangle

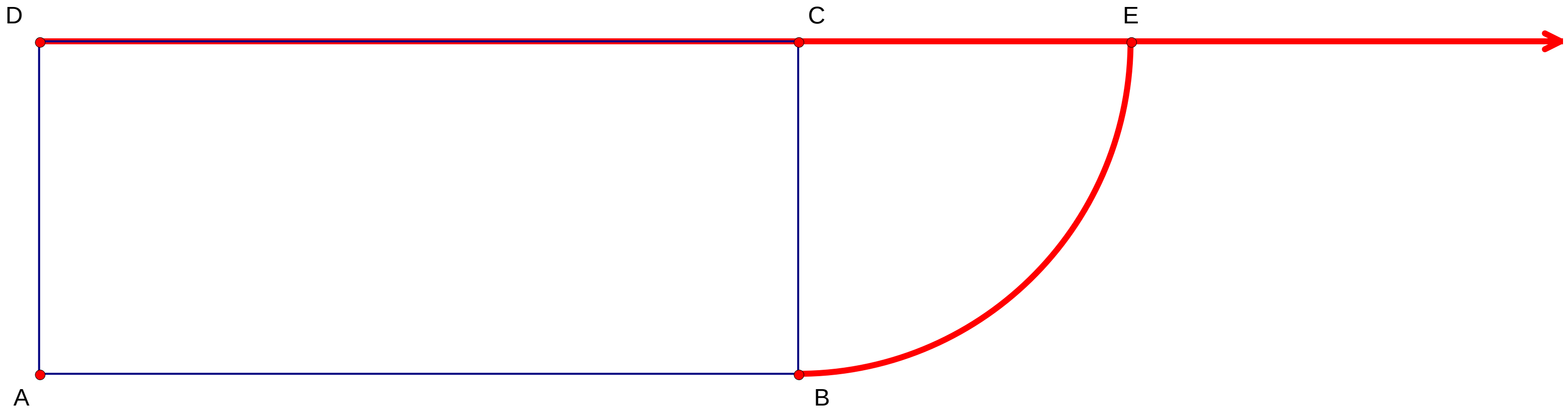
that is, making a square whose area is equal to the rectangle we start with

We start with a rectangle, ABCD.



Our task is to construct a square with area equal to the area of rectangle ABCD

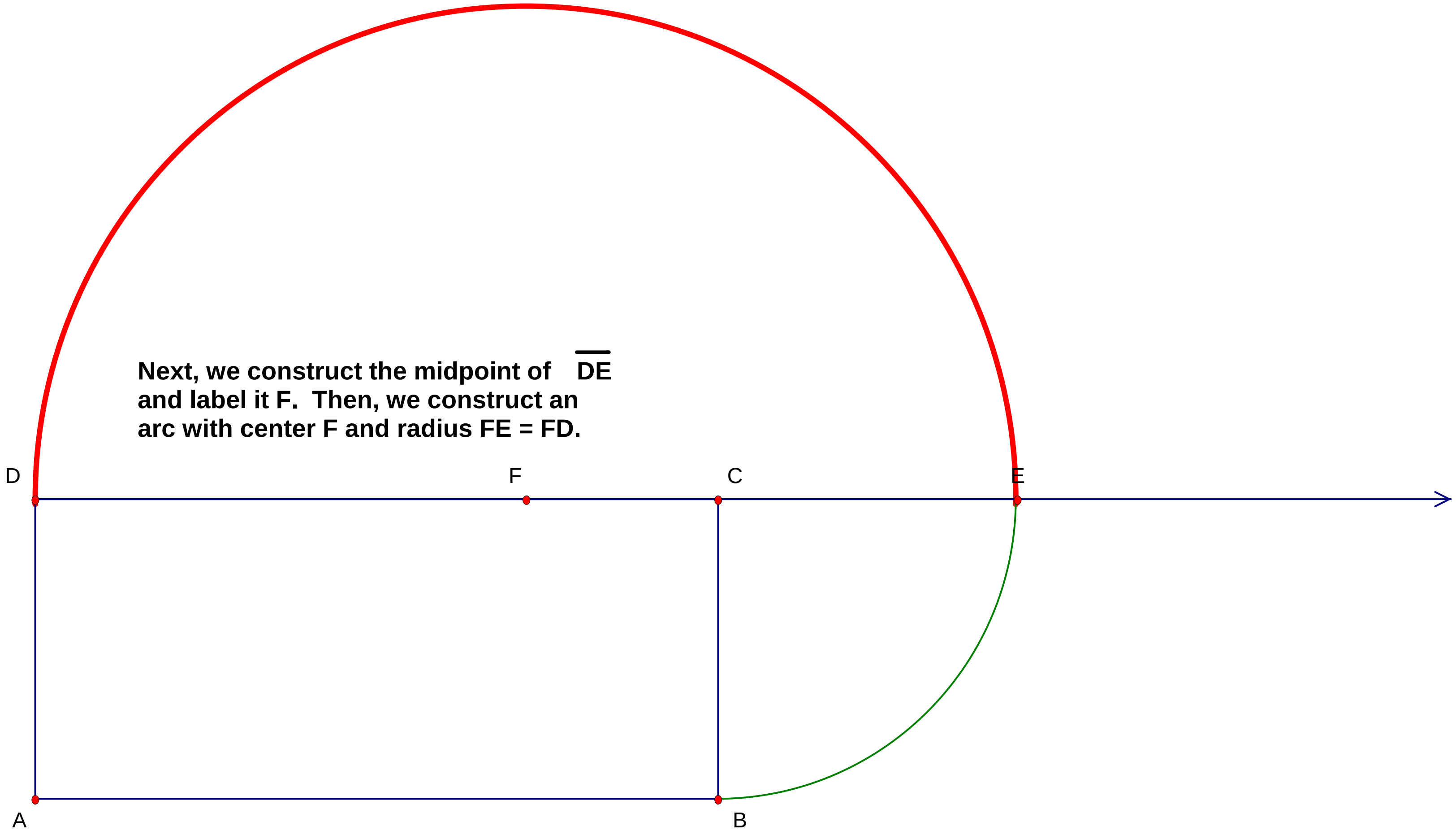
We begin by extending $\overline{DC}$ to form $\overrightarrow{DC}$.



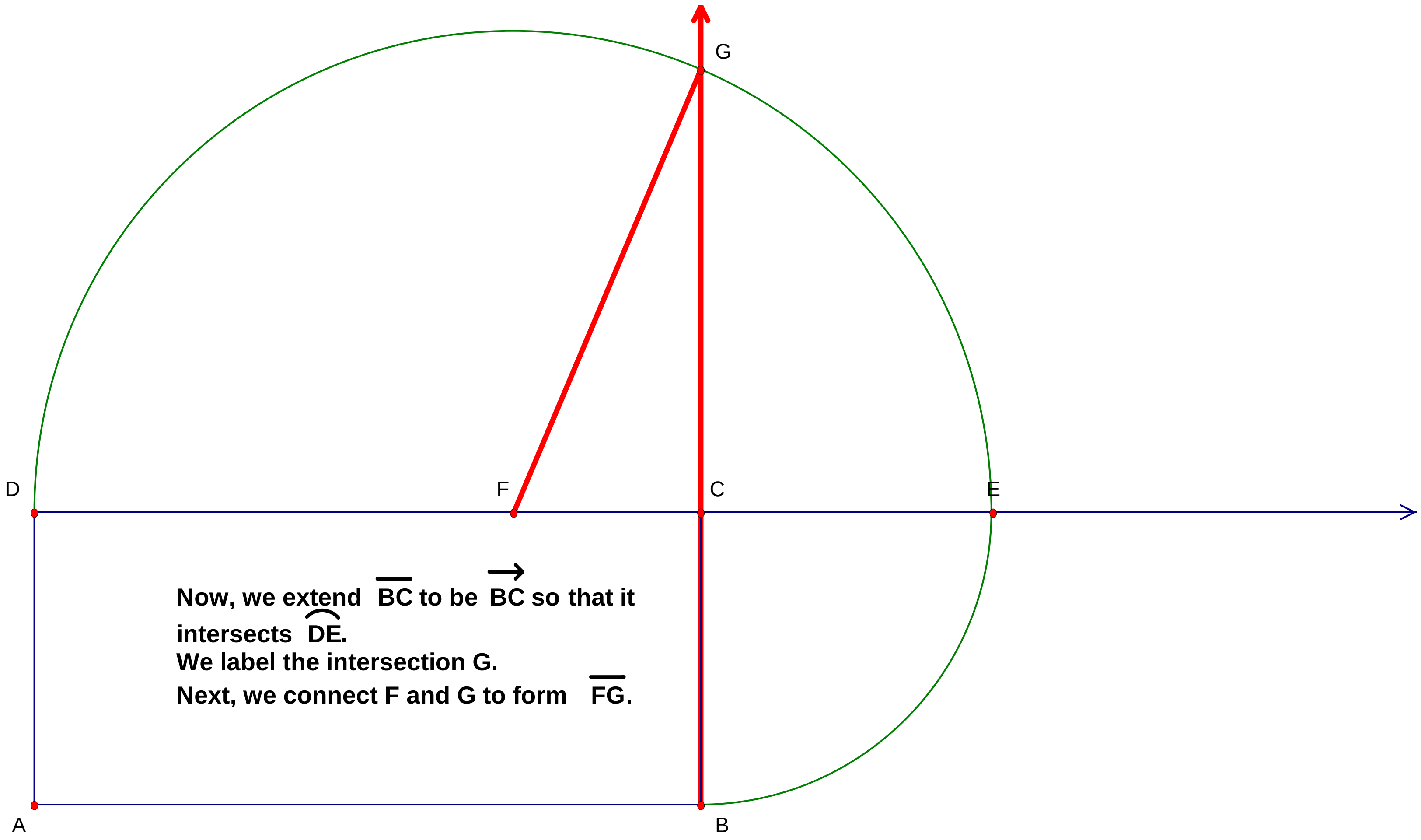
Then we draw an arc with center C and radius CB

from B until it intersects $\overrightarrow{DC}$.

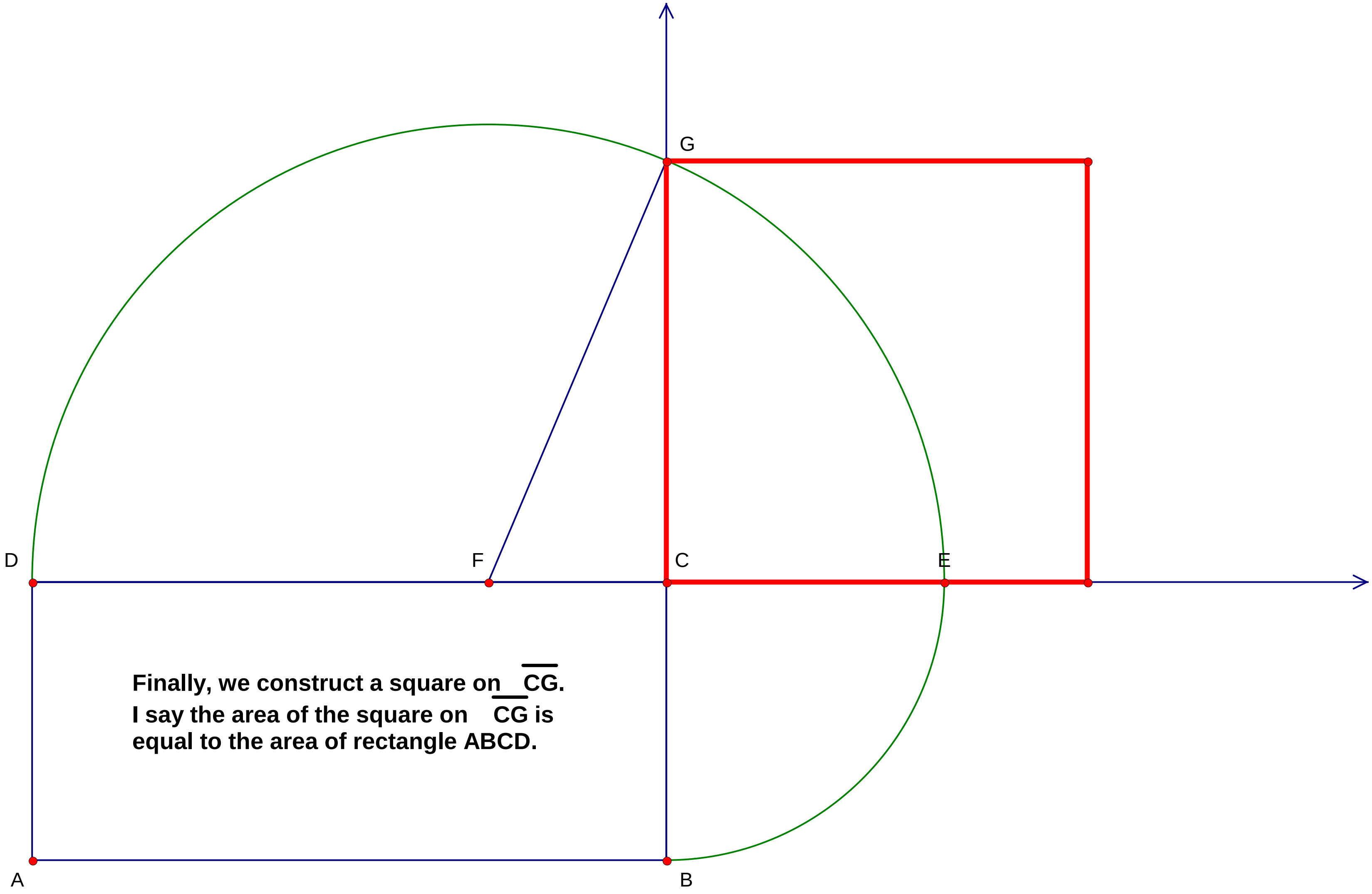
We label the intersection E.



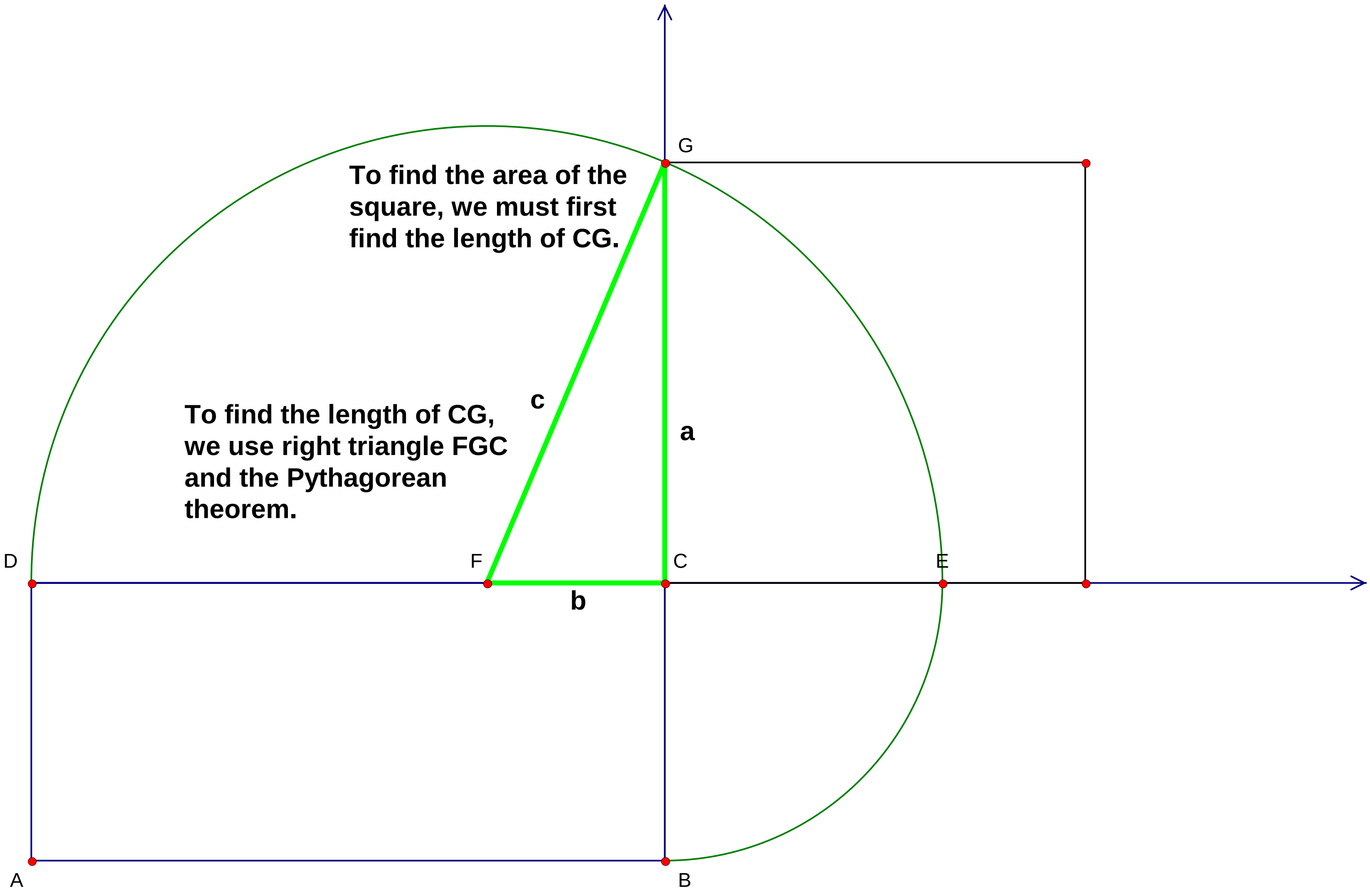
Next, we construct the midpoint of $\overline{DE}$ and label it F. Then, we construct an arc with center F and radius FE = FD.



Now, we extend $\overline{BC}$ to be $\overrightarrow{BC}$ so that it intersects $\overset{\frown}{DE}$.
We label the intersection G.
Next, we connect F and G to form $\overline{FG}$.

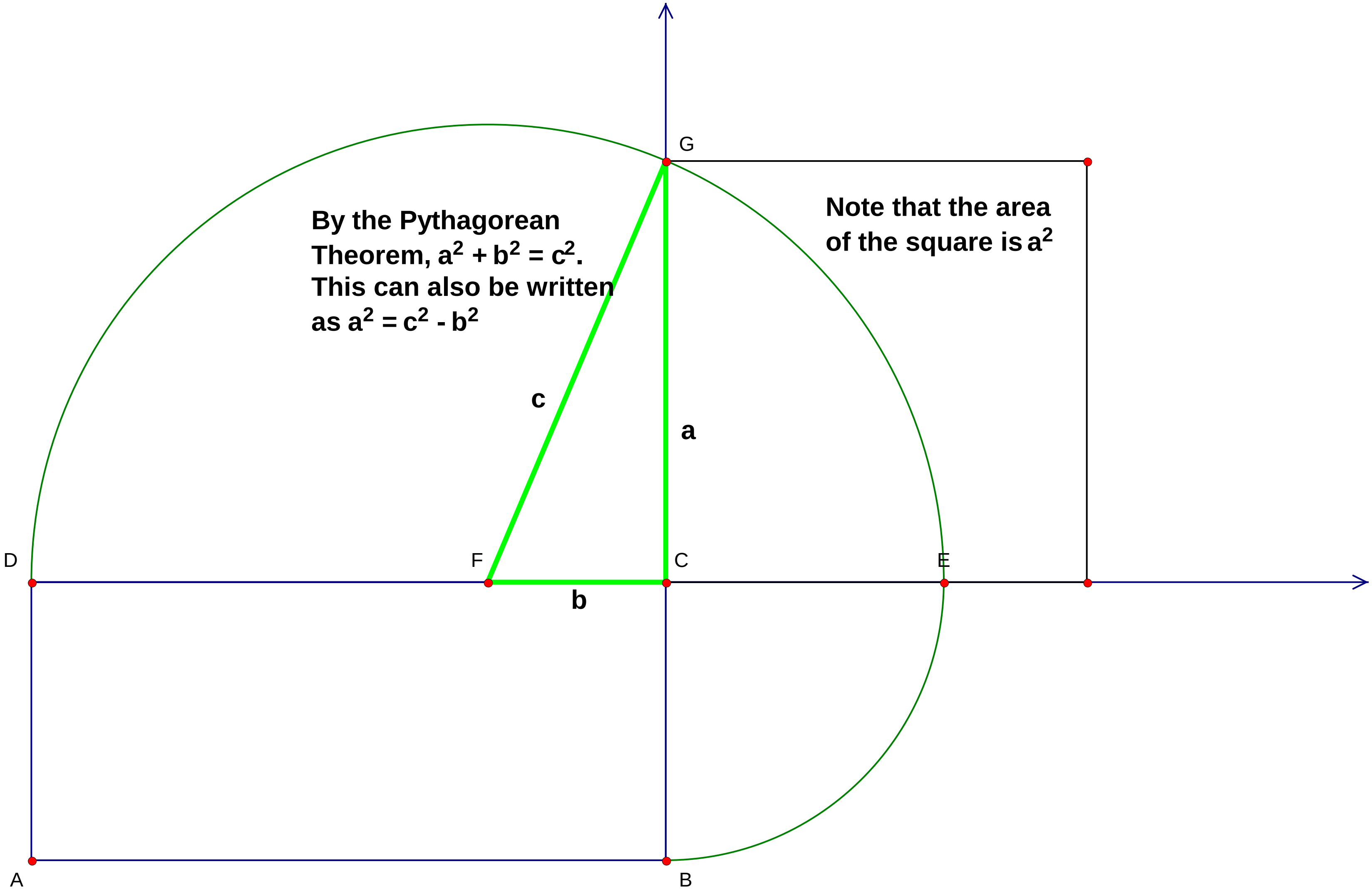


Finally, we construct a square on $\overline{CG}$.
I say the area of the square on $\overline{CG}$ is
equal to the area of rectangle ABCD.



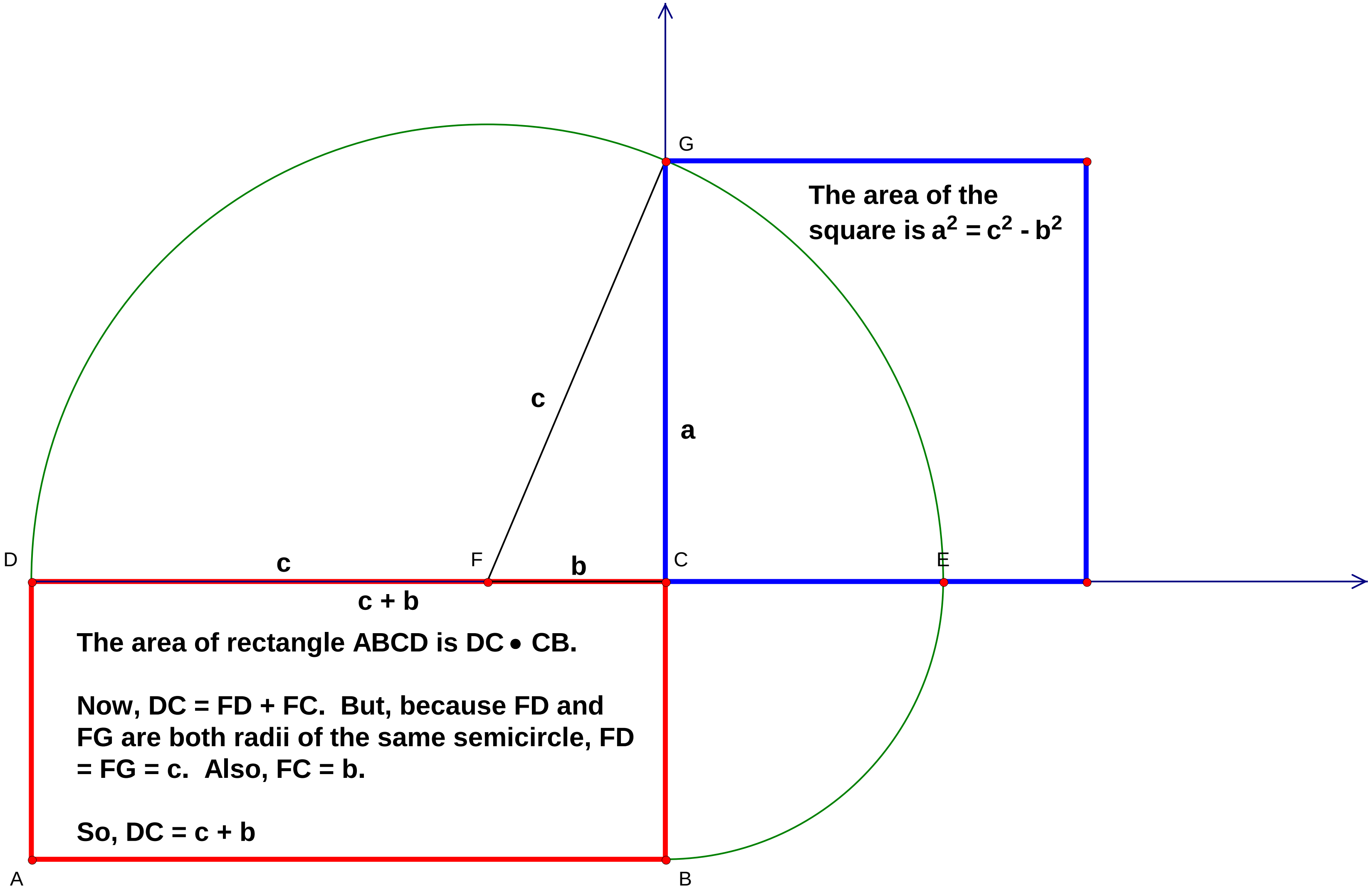
To find the area of the square, we must first find the length of CG.

To find the length of CG, we use right triangle FGC and the Pythagorean theorem.



By the Pythagorean Theorem, $a^2 + b^2 = c^2$. This can also be written as $a^2 = c^2 - b^2$

Note that the area of the square is a^2

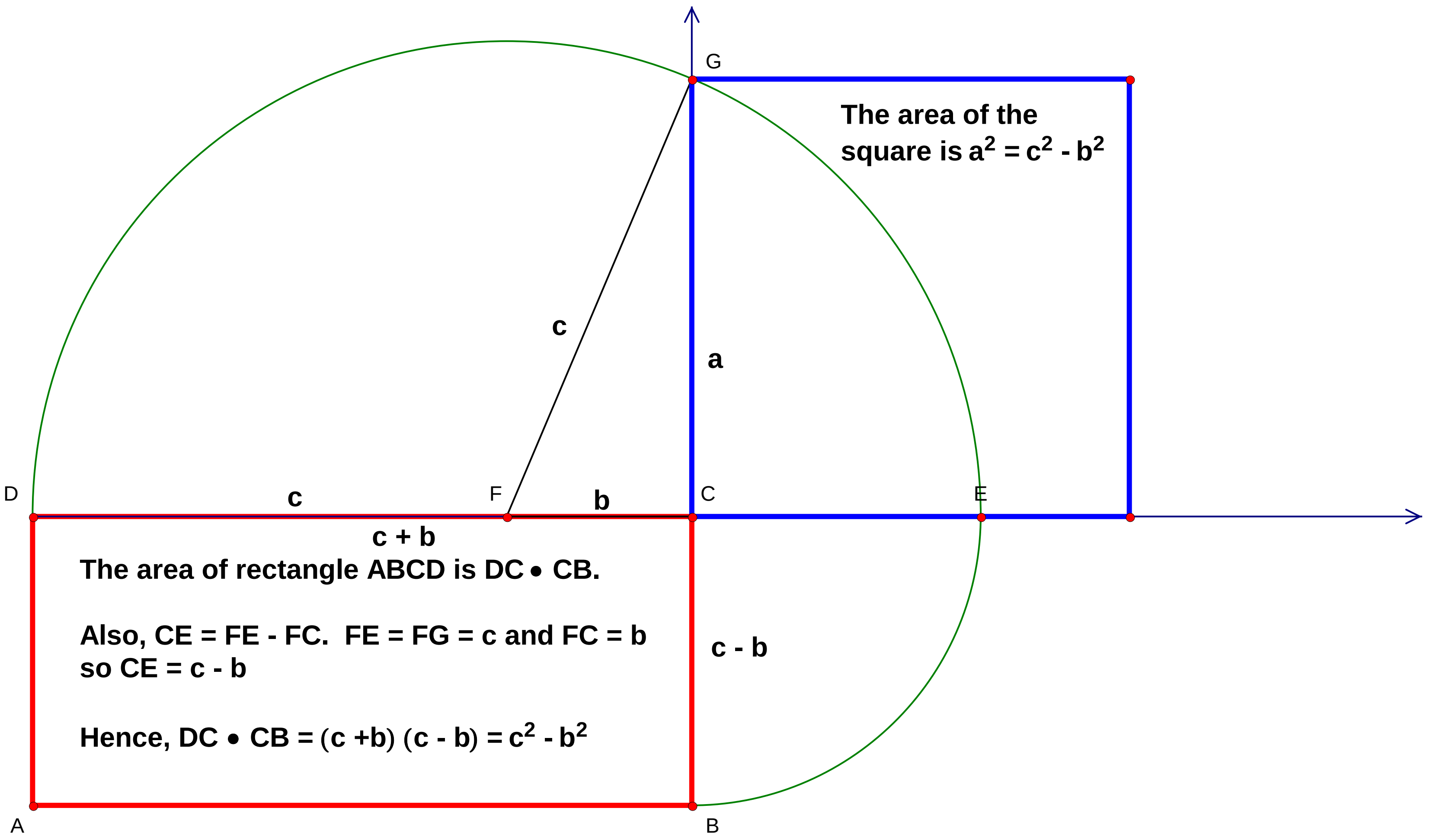


The area of the square is $a^2 = c^2 - b^2$

The area of rectangle ABCD is DC $\bullet$ CB.

Now, DC = FD + FC. But, because FD and FG are both radii of the same semicircle, FD = FG = c. Also, FC = b.

So, DC = c + b



The area of the square is $a^2 = c^2 - b^2$

The area of rectangle ABCD is DC • CB.

Also, CE = FE - FC. FE = FG = c and FC = b so CE = c - b

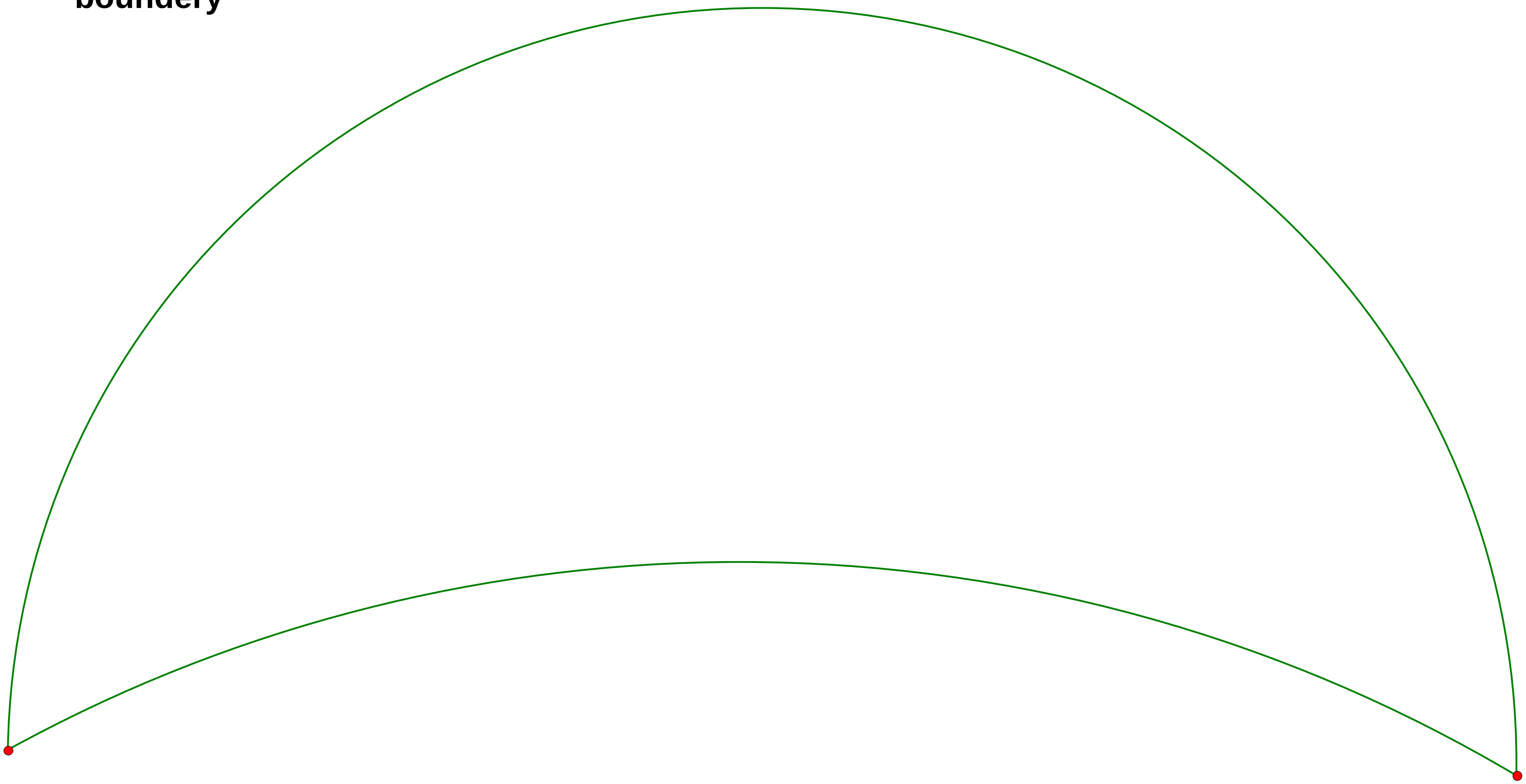
Hence, DC • CB = (c +b) (c - b) = $c^2 - b^2$

The area of the square and the area of the rectangle are the same. QED

After the rectangle is squared

- Triangles are quadrable because they are half as big as rectangles
- Polygons are quadrable because they can be cut up into triangles
- Finally, Hippocrates squared a lune, a moon shaped piece of a circle.

A lune is a part of a circle with a curved boundery



The quadrature of the lune requires:

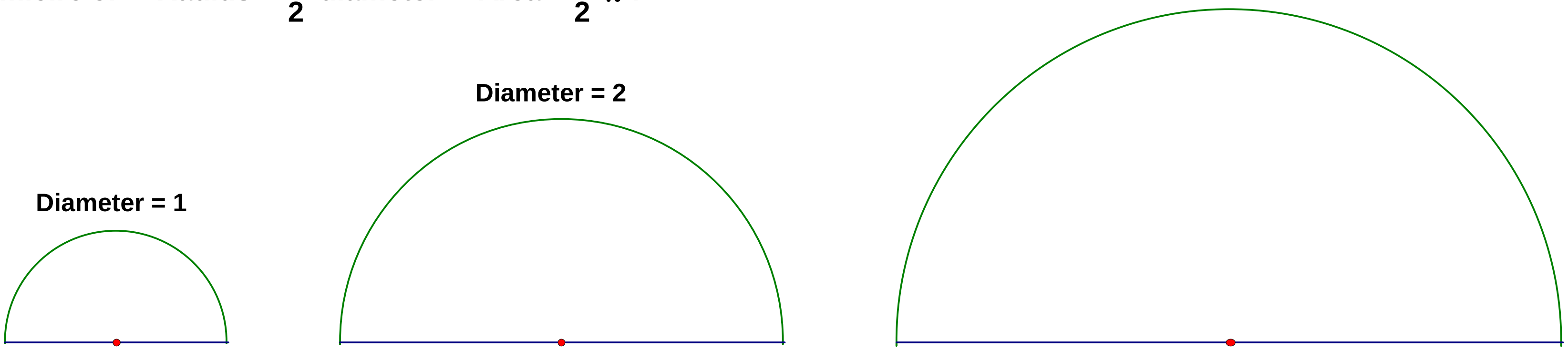
- The Pythagorean theorem
- An angle inscribed in a semicircle is right
- The areas of two semicircles are to each other as the squares on their diameters

$$\frac{Area\left(semicircle\,1\right)}{Area\left(semicircle\,2\right)}=\frac{d^2}{D^2}$$

This third requirement needs some explanation

A semicircle with twice the diameter has 4 times the area. A semicircle with triple the diameter has 9 times the area, and so on.

Semicircle: **Radius** $= \frac{1}{2}$ **diameter** **Area** $= \frac{1}{2} \pi r^2$



Diameter = 1

Diameter = 2

Diameter = 3

Area $= \frac{1}{8} \pi$

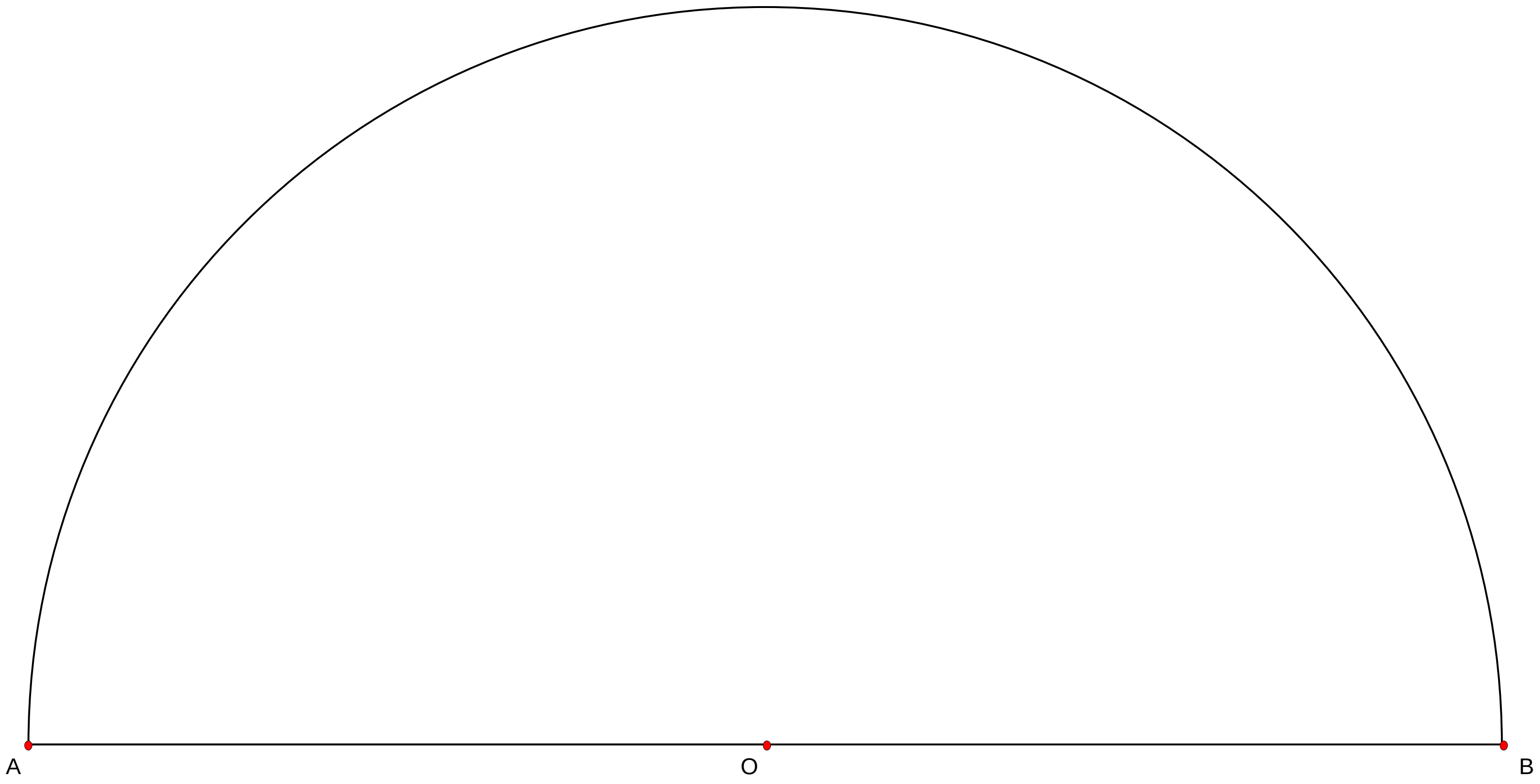
Area $= \frac{1}{2} \pi$

Area $= \frac{9}{8} \pi$

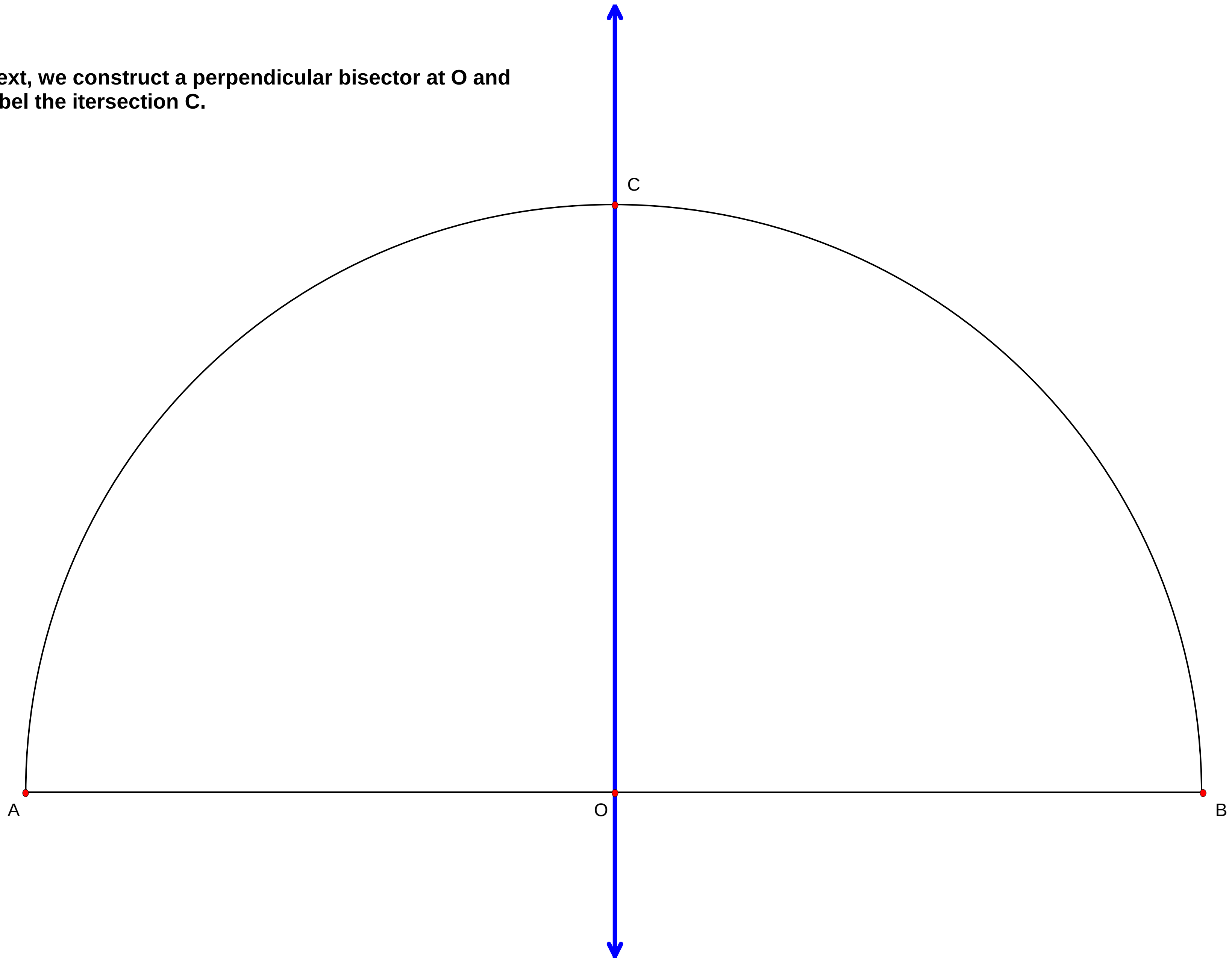
4 times as big

9 times as big

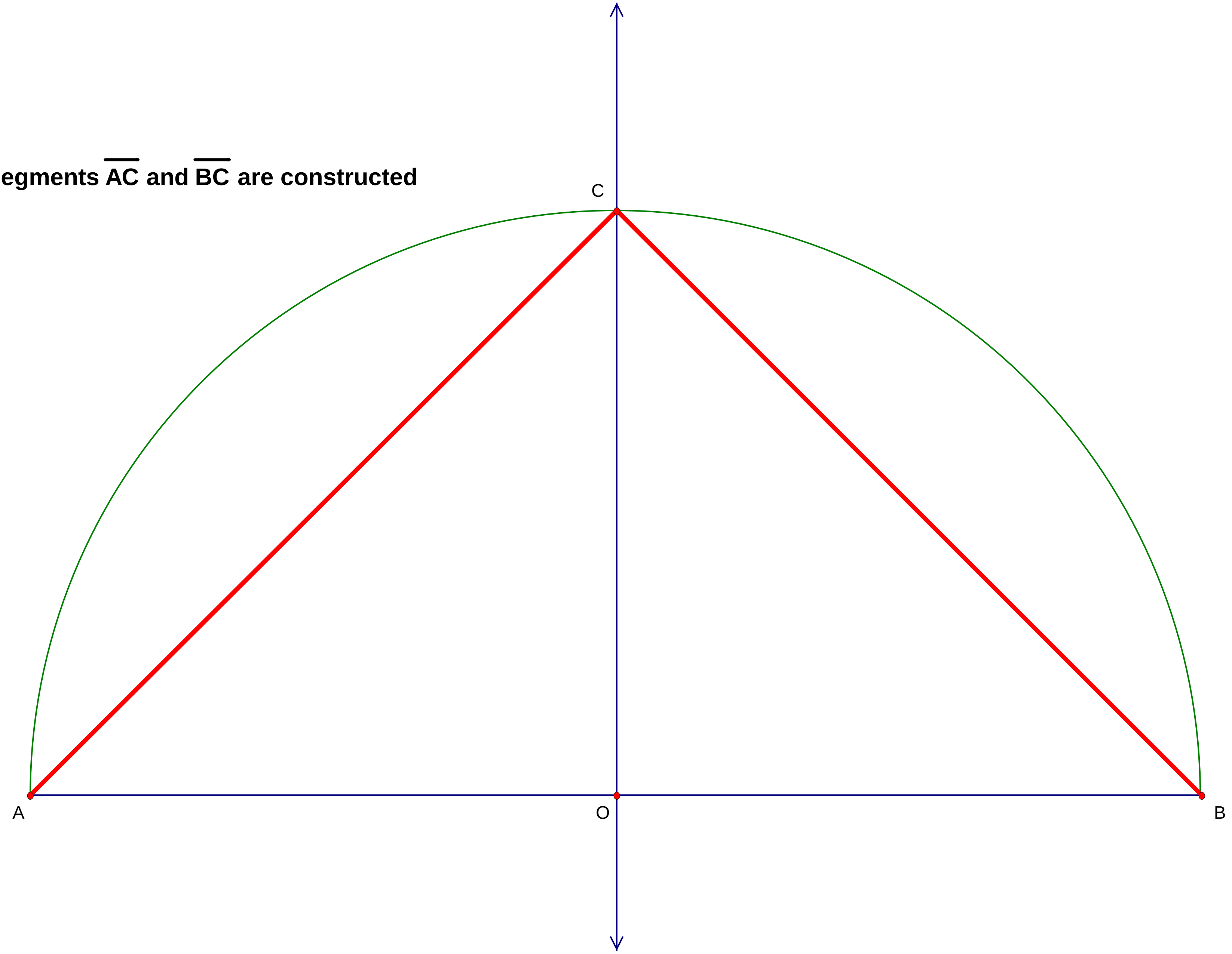
To square a lune, we first must construct a lune.
We start by constructing a semicircle on segment $\overline{AB}$.



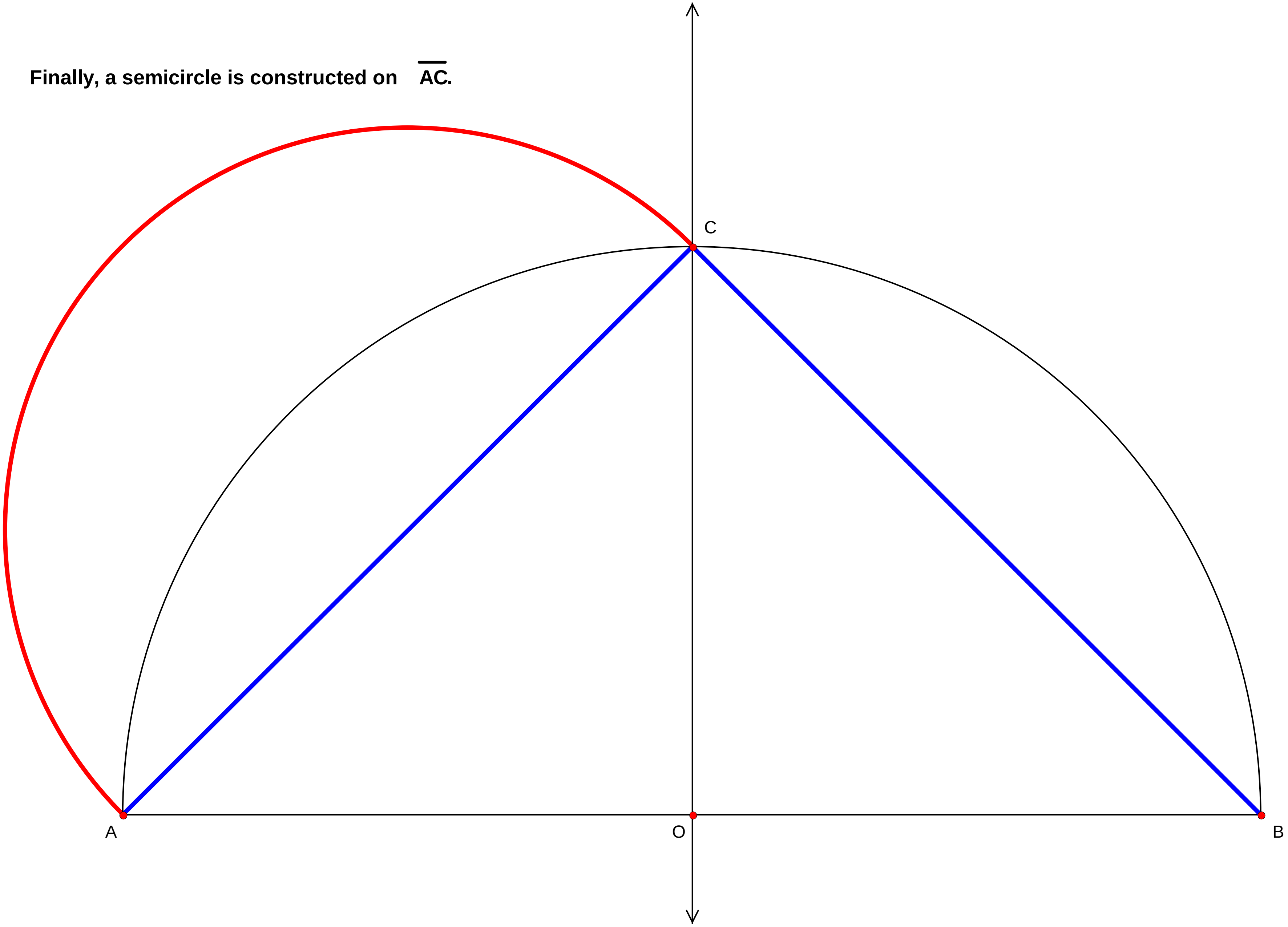
Next, we construct a perpendicular bisector at O and label the itersection C.

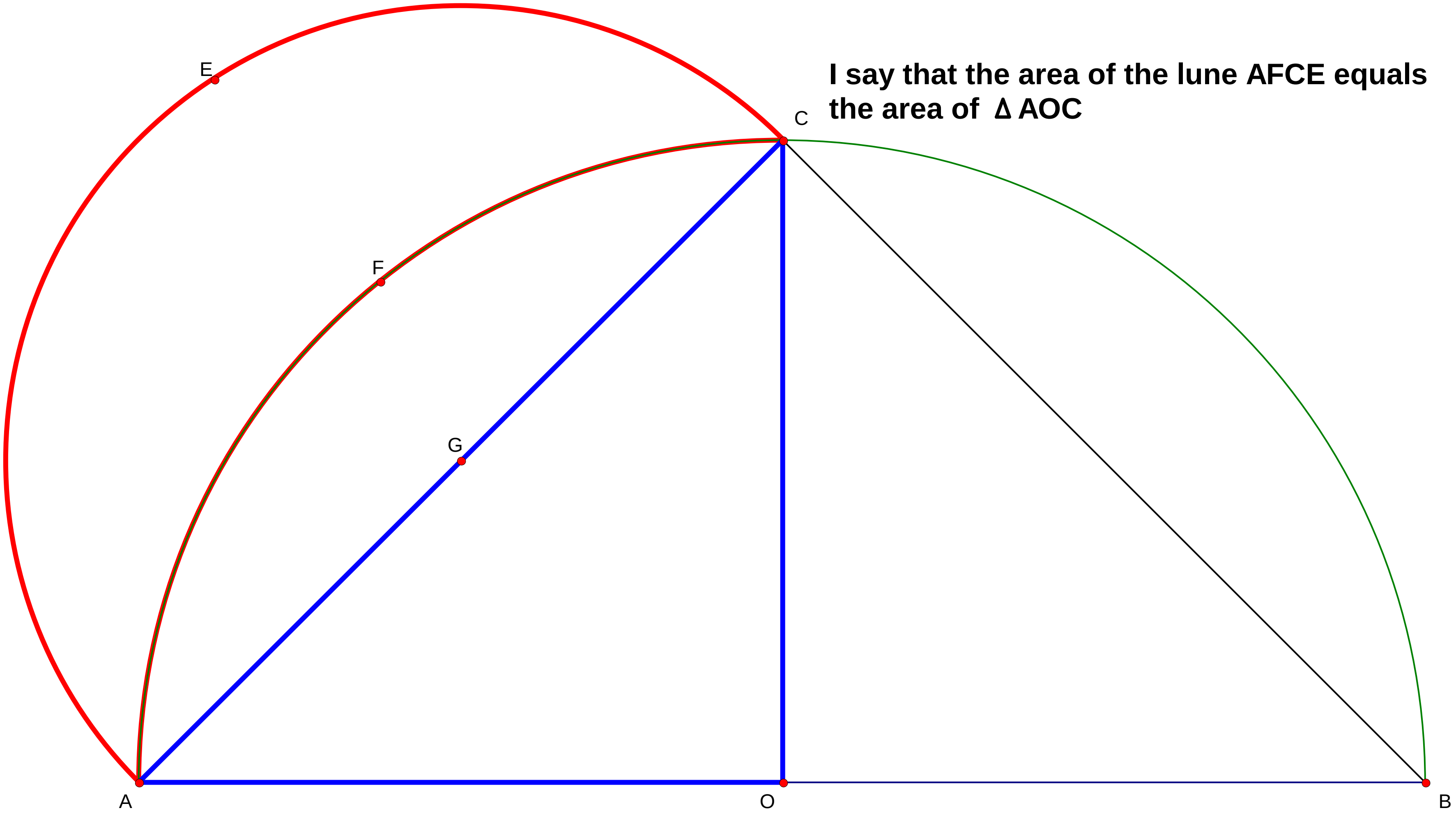


Then, segments $\overline{AC}$ and $\overline{BC}$ are constructed

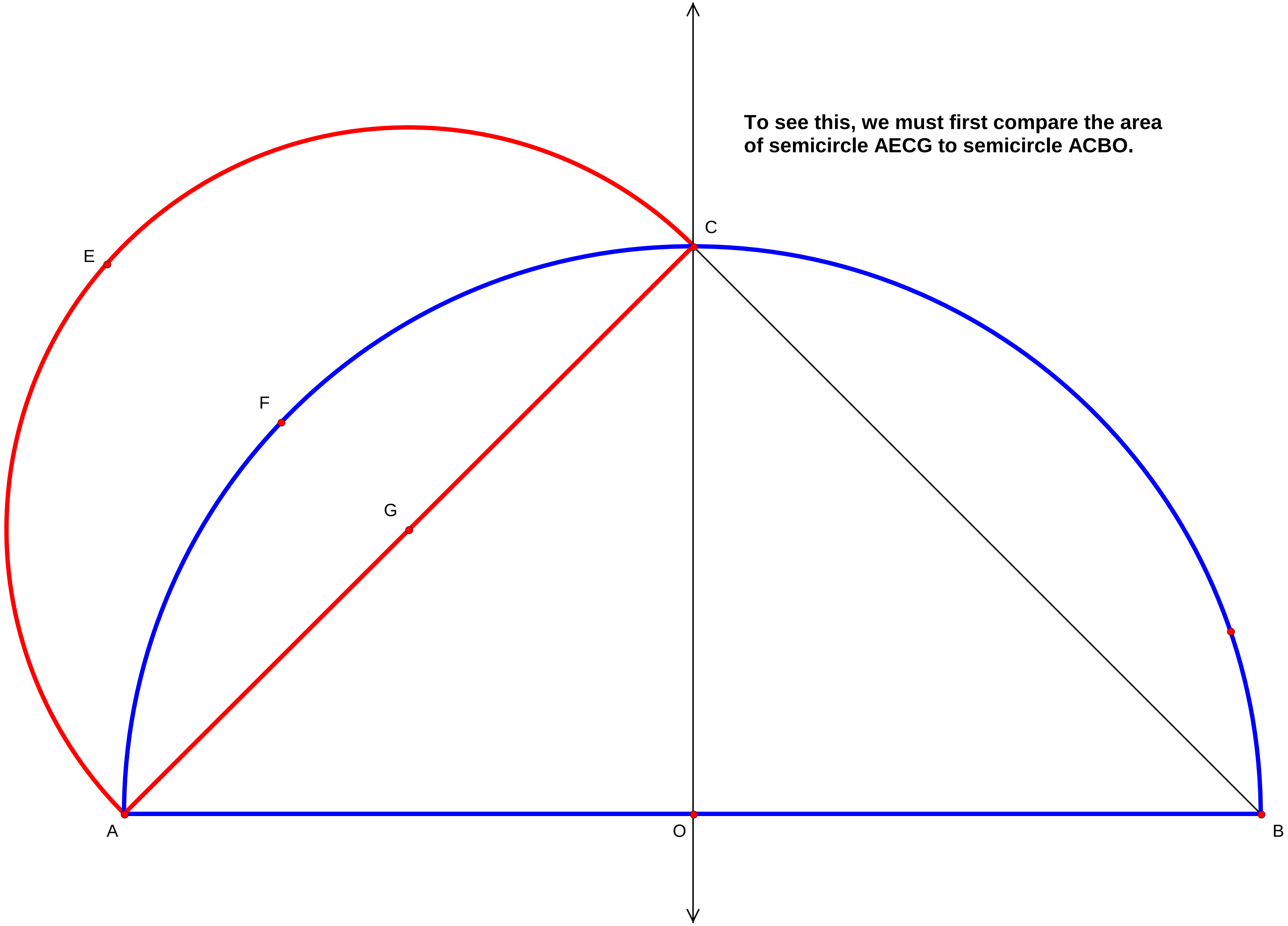


Finally, a semicircle is constructed on $\overline{AC}$.

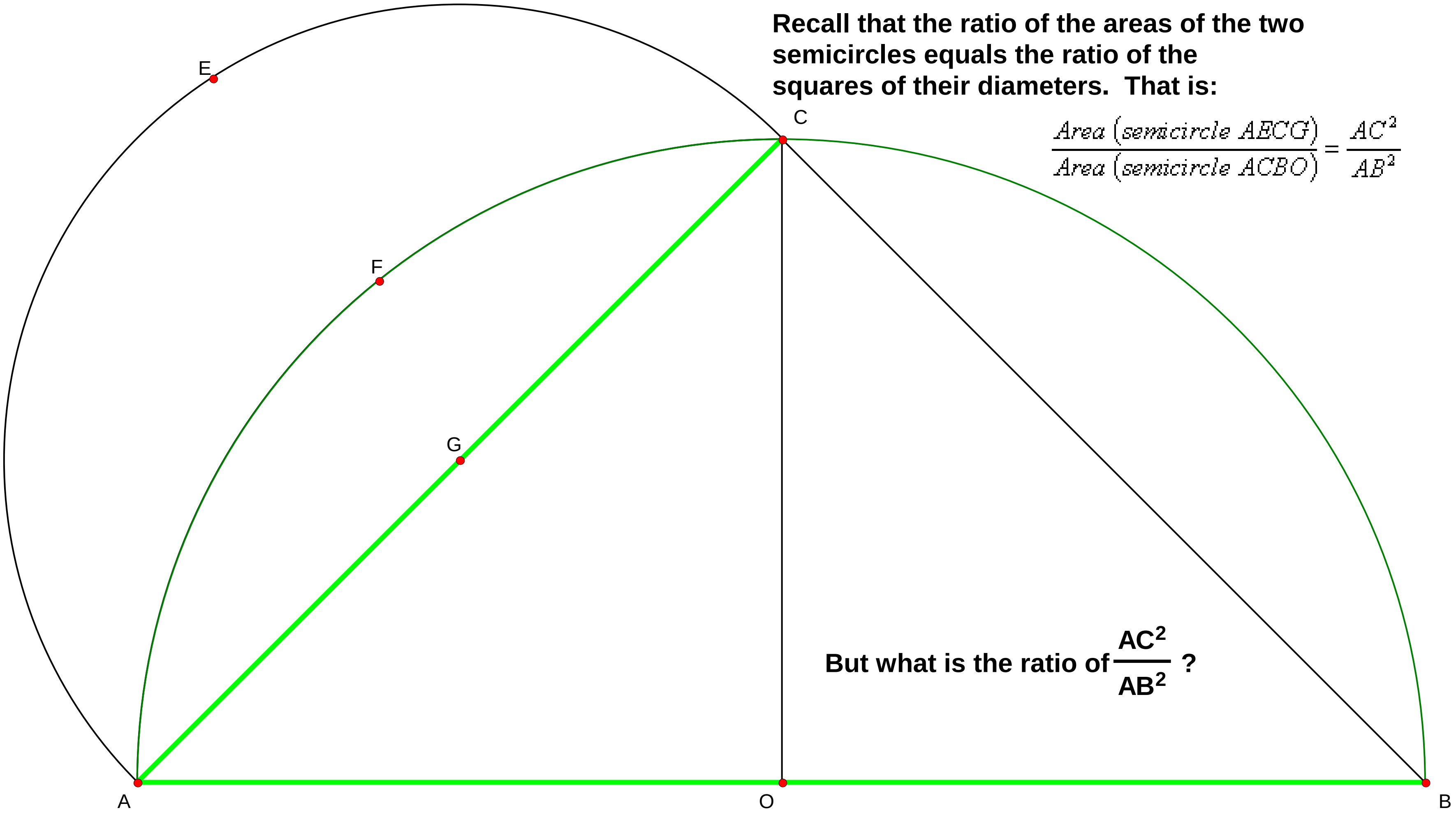




I say that the area of the lune AFCE equals the area of Δ AOC



To see this, we must first compare the area of semicircle AECG to semicircle ACBO.



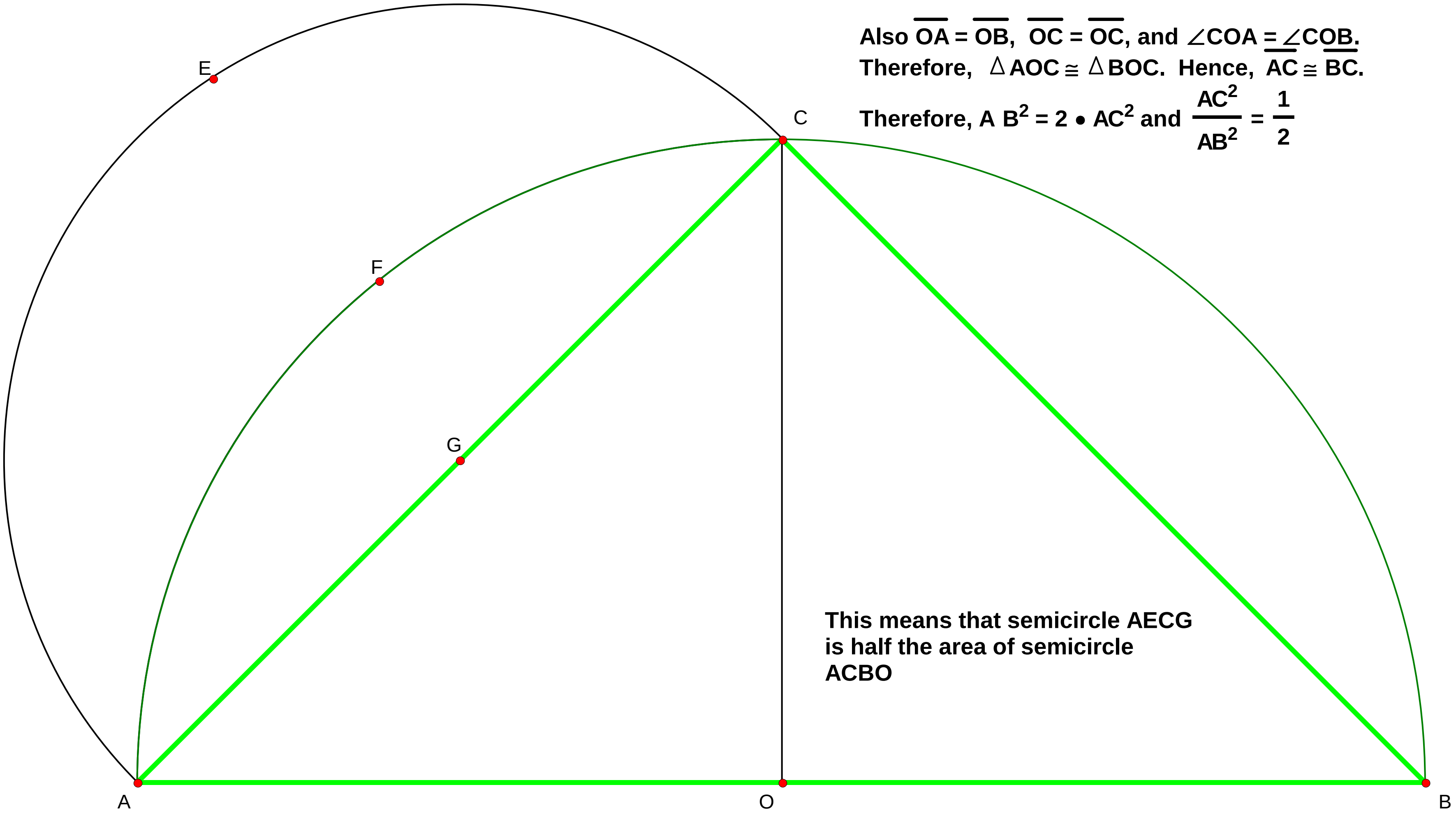
Recall that the ratio of the areas of the two semicircles equals the ratio of the squares of their diameters. That is:

$$\frac{Area\left(semicircle\ AECG\right)}{Area\left(semicircle\ ACBO\right)} = \frac{AC^2}{AB^2}$$

But what is the ratio of $\frac{AC^2}{AB^2}$?

Consider $\triangle ABC$. $AB^2 = AC^2 + BC^2$

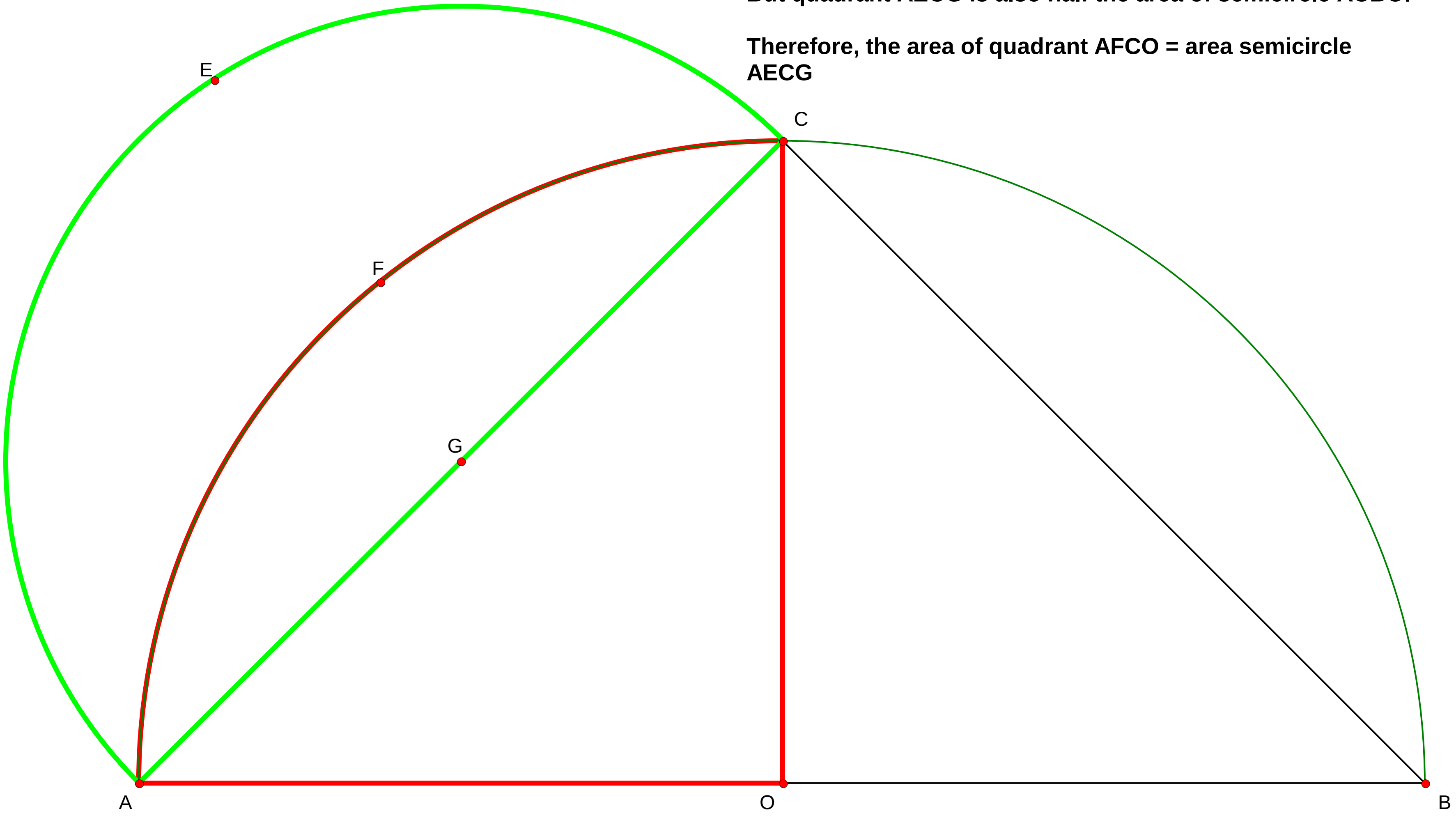
Also $\overline{OA} = \overline{OB}$, $\overline{OC} = \overline{OC}$, and $\angle COA = \angle COB$.
Therefore, $\triangle AOC \cong \triangle BOC$. Hence, $\overline{AC} \cong \overline{BC}$.
Therefore, $AB^2 = 2 \bullet AC^2$ and $\dfrac{AC^2}{AB^2} = \dfrac{1}{2}$



This means that semicircle AECG is half the area of semicircle ACBO

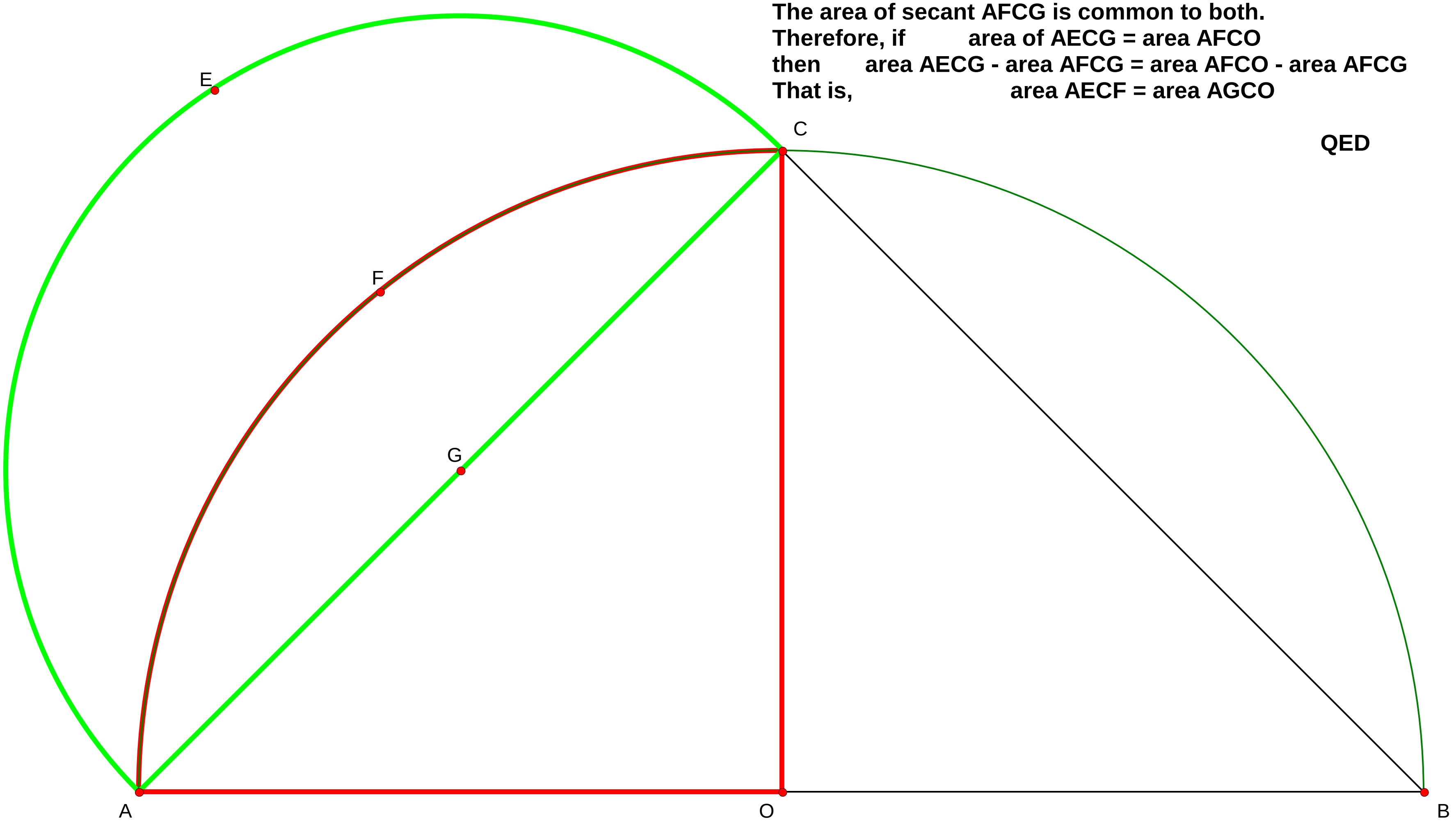
But quadrant AECG is also half the area of semicircle ACBO.

Therefore, the area of quadrant AFCO = area semicircle AECG



The area of secant AFCG is common to both.
Therefore, if area of AECG = area AFCO
then area AECG - area AFCG = area AFCO - area AFCG
That is, area AECF = area AGCO

QED



The Lune Was Squared!

- Hippocrates managed to square two other particular lunes.
- It seemed that the circle could also be squared, attempts were made for more than 2,000 years
- In 1771, Euler squared two other particular lunes.
- Now it really seemed like the circle was next.

But, in 2,000 years of trying

- No one was quite able to square the circle.
- Many claims were made but they all were proven flawed.
- Finally, in 1886, Lindeman proved that it couldn’t be done.
- However, the search had spurred great mathematical research along the way.